

Modulation Spaces and Nonlinear Evolution Equations

Michael Ruzhansky^a, Mitsuru Sugimoto^b, Baoxiang Wang^{c*}

February 28, 2013

Abstract

We survey some recent progress on modulation spaces and the well-posedness results for a class of nonlinear evolution equations by using the frequency-uniform localization techniques.

1 Definitions of modulation spaces

The modulation space was introduced by Feichtinger [15] in 1983. We denote by \mathfrak{S} the Schwartz space and by \mathfrak{S}' its dual space. Let f, g be Schwartz functions, and set

$$V_g f(x, \omega) = \int_{\mathbb{R}^n} e^{-it\omega} \overline{g(t-x)} f(t) dt.$$

$V_g f$ is said to be the short-time Fourier transform of f . In the following, if there is no explanation, we will always assume that

$$-\infty < s < \infty, \quad 0 < p, q \leq \infty, \quad \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}.$$

The norm on modulation spaces is given by

$$\|f\|_{M_{p,q}^s}^o = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_g f(x, \omega)|^p dx \right)^{q/p} \langle \omega \rangle^{sq} d\omega \right)^{1/q}, \quad (1.1)$$

with a natural modification for $p, q = \infty$. Applying the frequency-uniform localization techniques, one can give an equivalent norm on modulation spaces. Let Q_k be the unit cube with its center at k , $\{Q_k\}_{k \in \mathbb{Z}^n}$ constitutes a decomposition of \mathbb{R}^n . Such kind of

^aDepartment of Mathematics, Imperial College London, 180 Queens Gate, London SW7 2AZ United Kingdom, Email: m.ruzhansky@imperial.ac.uk, ^bGraduate School of Mathematics, Nagoya University, Furocho, Chikusa-ku, Nagoya 464-8602 Japan, Email: sugimoto@math.nagoya-u.ac.jp, ^cLMAM, School of Mathematical Sciences, Peking University, and Beijing International Center of Mathematical Research, Beijing 100871, PR of China, Email: wbx@math.pku.edu.cn

decomposition goes back to the work of N. Wiener [57], and we say that it is the Wiener decomposition of \mathbb{R}^n . We can roughly write

$$\square_k \sim \mathfrak{F}^{-1} \chi_{Q_k} \mathfrak{F}, \quad k \in \mathbb{Z}^n, \quad (1.2)$$

where χ_E denotes the characteristic function on the set E , and \mathfrak{F} is the Fourier transform. Since Q_k is a translation of Q_0 , \square_k ($k \in \mathbb{Z}^n$) have the same localized structures in the frequency space, which are said to be the *frequency-uniform decomposition operators*. Similar to Besov spaces, one can use $\{\square_k\}_{k \in \mathbb{Z}^n}$ and $\ell^q(L^p)$ to generate a class of function spaces, so called modulation spaces for which the norm is defined by

$$\|f\|_{M_{p,q}^s} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_p^q \right)^{1/q},$$

with the natural modifications for $p, q = \infty$. We now give an exact definition on frequency-uniform decomposition operators. Since χ_{Q_k} can not be differentiated, it is convenient to replace χ_{Q_k} in (1.2) by a smooth cut-off function. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$, $\rho : \mathbb{R}^n \rightarrow [0, 1]$ be a smooth function verifying $\rho(\xi) = 1$ for $|\xi|_\infty \leq 1/2$ and $\rho(\xi) = 0$ for $|\xi|_\infty \geq 1$ ¹. Let ρ_k be a translation of ρ ,

$$\rho_k(\xi) = \rho(\xi - k), \quad k \in \mathbb{Z}^n. \quad (1.3)$$

We see that $\rho_k(\xi) = 1$ in Q_k and so, $\sum_{k \in \mathbb{Z}^n} \rho_k(\xi) \geq 1$ for all $\xi \in \mathbb{R}^n$. Denote

$$\sigma_k(\xi) = \rho_k(\xi) \left(\sum_{\ell \in \mathbb{Z}^n} \rho_\ell(\xi) \right)^{-1}, \quad k \in \mathbb{Z}^n. \quad (1.4)$$

Then we have

$$\begin{cases} |\sigma_k(\xi)| \geq c, & \forall \xi \in Q_k, \\ \text{supp } \sigma_k \subset \{\xi : |\xi - k|_\infty \leq 1\}, \\ \sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) \equiv 1, & \forall \xi \in \mathbb{R}^n, \\ |D^\alpha \sigma_k(\xi)| \leq C_{|\alpha|}, & \forall \xi \in \mathbb{R}^n, \alpha \in (\mathbb{N} \cup \{0\})^n. \end{cases} \quad (1.5)$$

Hence, the set

$$\Upsilon_n = \{\{\sigma_k\}_{k \in \mathbb{Z}^n} : \{\sigma_k\}_{k \in \mathbb{Z}^n} \text{ satisfies (1.5)}\} \quad (1.6)$$

is non-void. If there is no confusion, in the sequel we will write $\Upsilon = \Upsilon_n$. Let $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$. The frequency-uniform decomposition operators can be exactly defined by

$$\square_k := \mathfrak{F}^{-1} \sigma_k \mathfrak{F}, \quad k \in \mathbb{Z}^n. \quad (1.7)$$

For simplicity, we write $M_{p,q}^0 = M_{p,q}$. The space $M_{p,q}^s$ is said to be the modulation space. One can prove that $\|\cdot\|_{M_{p,q}^s}^\circ$ and $\|\cdot\|_{M_{p,q}^s}$ are equivalent norms; cf. [15] for a proof on modulation spaces defined on Abelian groups, and [55] for a straightforward proof.

¹For $\xi = (\xi_1, \dots, \xi_n)$, $|\xi|_\infty := \max_{i=1, \dots, n} |\xi_i|$.

2 Questions on the Schrödinger equation

Let $S(t) = e^{it\Delta}$ be the Schrödinger semi-group. It is known that $S(t) = e^{it\Delta} : L^p \rightarrow L^p$ is bounded if and only if $p = 2$. Indeed, one can show that

$$\sup_{\varphi \in \mathfrak{S} \setminus \{0\}} \frac{\|S(t)\varphi\|_{L^p(\mathbb{R}^n)}}{\|\varphi\|_{L^p(\mathbb{R}^n)}} = \infty, \quad p \neq 2, \quad (2.1)$$

by taking $\varphi = e^{-(a+ib)|x|^2}$. So, a natural question is the following

Question 2.1 *Is there any Banach function space X satisfying*

$$\sup_{\varphi \in X \setminus \{0\}} \frac{\|S(t)\varphi\|_{X(\mathbb{R}^n)}}{\|\varphi\|_{X(\mathbb{R}^n)}} < \infty? \quad (2.2)$$

Recall that (cf. [5, 6])

$$\|S(t)f\|_p \leq C|t|^{-n(1/2-1/p)}\|f\|_{p'}, \quad (2.3)$$

where $2 \leq p \leq \infty$, $1/p + 1/p' = 1$. Estimate (2.3) contains singularity at $t = 0$, and another natural question is

Question 2.2 *Is there any Banach function space X satisfying the following truncated decay*

$$\|S(t)\varphi\|_{X(\mathbb{R}^n)} \leq C(1 + |t|)^{-\delta(X)}\|\varphi\|_{X'(\mathbb{R}^n)}? \quad (2.4)$$

where $\delta > 0$, X' denotes the dual space of X , X and X' have the same regularity.

Consider the Cauchy problem for the nonlinear Schrödinger equation (NLS)

$$iu_t + \Delta u = |u|^{2\kappa}u, \quad u(0, x) = u_0(x), \quad \kappa \in \mathbb{N}. \quad (2.5)$$

It is known that if u is a solution of (2.5), so is $u_\mu(t, x) = \mu^{1/\kappa}u(\mu^2t, \mu x)$ with the initial datum $\mu^{1/\kappa}u_0(\mu x)$. Taking notice of

$$\|u_\mu(0)\|_{\dot{H}^s} = \mu^{s-n/2+1/\kappa}\|u_0\|_{\dot{H}^s}, \quad \mu > 0,$$

one sees that $s = s_\kappa := n/2 - 1/\kappa$ is the index such that $\|u_\mu(0)\|_{\dot{H}^s}$ is invariant for all $\mu > 0$. \dot{H}^{s_κ} is said to be a critical space for NLS (2.5); cf. [6]. Up to now, we can solve (2.5) in H^s for $s \geq s_\kappa \geq 0$, which correspond to the critical and sub-critical cases in H^s . However, we cannot solve (2.5) in H^s in the case $s < s_\kappa$, where H^s is said to be the super-critical space. Our question is

Question 2.3 *Let $0 \leq s < s_\kappa$. Are there any initial data $u_0 \in H^s$ so that NLS (2.5) is still locally and globally well posed?*

In order to answer the above questions, modulation spaces play important roles. Roughly speaking, Schrödinger semi-group is bounded in modulation spaces and it satisfies a truncated decay in modulation spaces. Moreover, the modulation space $M_{2,1}$ enjoys lower derivative regularity and we can solve NLS (2.5) in $M_{2,1}$ for all $\kappa \in \mathbb{N}$. $M_{2,1}$ contains a class of data which are out of the control of H^{s_κ} if $s_\kappa > 0$.

Finally, we mention that all the questions above make sense for other equations as well. For example, the propagators $W_\pm(t) = e^{\pm it\sqrt{-\Delta}}$ for the wave equation display the loss of regularity for $p \neq 2$. More precisely, we have that for $1 < p < \infty$ and $t > 0$, the operator $W_+(t) : W^{s,p}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is (locally) bounded if and only if $s \geq (n-1)|1/p - 1/2|$. The same holds for $W_-(t)$, and Question 2.1 is valid. Also, a variant (2.3) holds², so Question 2.2 is also valid, as well as the corresponding version of Question 2.3.

We note that the dependence of estimates and large time asymptotics in Sobolev spaces for the propagator $e^{ita(D)}$ on $a(\xi)$ is known, especially in the first order case. For example, let $a \in C^\infty(\mathbb{R}^n \setminus 0)$ be a real-valued positively homogeneous function, satisfying $\nabla a(\xi) \neq 0$ for all $\xi \neq 0$. If we denote

$$k := \max_{|\xi|=1} \text{rank} \nabla_\xi^2 a(\xi),$$

then there is a loss of derivatives in L^p depending on k . More precisely, as a special case of estimates for Fourier integral operators for $1 < p < \infty$, it was shown in [37] (see also [38]) that if $e^{ia(D)} : W^{s,p} \rightarrow L^p$ is locally bounded then $s \geq k|1/p - 1/2|$. The dispersive estimates³

$$\|e^{ita(D)} f\|_p \leq C|t|^{-\kappa(1/2-1/p)} \|f\|_{p'},$$

where $2 \leq p \leq \infty$, $1/p + 1/p' = 1$, depend on the convexity of the level sets $\Sigma = \{\xi : a(\xi) = 1\}$ as well as on the maximal orders of contacts between Σ and its tangent lines. For a comprehensive analysis of such types of estimates and their applications to the Strichartz estimates we refer to [39].

3 Some results on modulation spaces

Roughly speaking, frequency-uniform decomposition operators combined with function spaces $\ell^q(L^p)$ produce modulation spaces. During the past thirty years, the importance of the frequency-uniform decomposition in applications seems to be not fully recognized, and it is even not mentioned in Gröchenig's famous book [18]. However, from PDE point of view, the combination of frequency-uniform decomposition operators and Banach function spaces $\ell^q(X(\mathbb{R}^n))$ ⁴ seems to be important in making nonlinear estimates, which contains an automatic decomposition on high-low frequencies.

²with n replaced by $n-1$.

³or rather the value of κ .

⁴ X is a Banach function space defined in \mathbb{R}^n .

3.1 Basic properties on modulation spaces

Proposition 3.1 (Completeness) *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.*

- (1) *$M_{p,q}^s$ is a (quasi-) Banach space. Moreover, if $1 \leq p, q \leq \infty$, then $M_{p,q}^s$ is a Banach space.*
- (2) *$\mathfrak{S}(\mathbb{R}^n) \subset M_{p,q}^s \subset \mathfrak{S}'(\mathbb{R}^n)$.*
- (3) *Let $0 < p, q < \infty$, then $\mathfrak{S}(\mathbb{R}^n)$ is dense in $M_{p,q}^s$.*

Proposition 3.2 (Equivalent norm) *Let $\{\sigma_k\}_{k \in \mathbb{Z}^n}, \{\varphi_k\}_{k \in \mathbb{Z}^n} \in \Upsilon$. Then $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ and $\{\varphi_k\}_{k \in \mathbb{Z}^n}$ generate equivalent norms on $M_{p,q}^s$.*

Proposition 3.2 indicates that one can choose $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$ according to our requirement. In applications of PDE, it is convenient to use the following $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$. Let $\{\eta_k\}_{k \in \mathbb{Z}} \in \Upsilon_1$, we denote

$$\sigma_k(\xi) := \eta_{k_1}(\xi_1) \dots \eta_{k_n}(\xi_n), \quad (3.1)$$

then we have $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$. the above $\sigma_k(\xi)$ realizes the separation of different variables.

Proposition 3.3 (Embedding) *Let $s_1, s_2 \in \mathbb{R}$ and $0 < p_1, p_2, q_1, q_2 \leq \infty$.*

- (1) *If $s_2 \leq s_1, p_1 \leq p_2$ and $q_1 \leq q_2$, then $M_{p_1, q_1}^{s_1} \subset M_{p_2, q_2}^{s_2}$.*
- (2) *If $q_2 < q_1$ and $s_1 - s_2 > n/q_2 - n/q_1$, then $M_{p, q_1}^{s_1} \subset M_{p, q_2}^{s_2}$.*

Proposition 3.4 (Dual space) *Let $s \in \mathbb{R}$ and $0 < p, q < \infty$. If $p \geq 1$, we denote $1/p + 1/p' = 1$; If $0 < p < 1$, we write $p' = \infty$. Then*

$$(M_{p,q}^s)^* = M_{p',q'}^{-s}. \quad (3.2)$$

If $p \geq 1$, Proposition 3.4 is similar to that of Besov spaces, however, if $0 < p < 1$, the result is quite different from that of Besov spaces. The details of the proof of Proposition 3.4 can be found in [55] by following the proof of the relevant result in Besov spaces.

Remark 3.5 *If $p, q \in [1, \infty]$, Propositions 3.1 and 3.4 were obtained by Feichtinger [15]. In [55, 56], the cases $0 < p < 1$ and $0 < q < 1$ were considered.*

Soon after the work [56], Kobayashi [29] independently defined $M_{p,q}$ for all $0 < p, q \leq \infty$ and obtained Proposition 3.1. Almost at the same time as [55], Kobayashi [30] discussed the dual space of $M_{p,q}$ and obtained partial results of Proposition 3.4: if $0 < p < 1$ or $1 < q < \infty$, he obtained $M_{p',q'} \subset (M_{p,q})^* \subset M_{\infty,\infty}$. For the other cases, he showed $(M_{p,q})^* = M_{p',q'}$. Recently, by using the molecular decomposition techniques of modulation spaces, Kobayashi and Sawano [31] reconsidered the dual space of $M_{p,q}^s$ and they also obtained the result of Proposition 3.4. It is worth to mention that Triebel [50] introduced a class of generalized modulation spaces for all indices $0 < p, q \leq \infty$, however, those spaces have no complete norms, which seems harder to use in the study of PDEs.

3.2 Inclusions between Besov and modulation spaces

From the definitions, we see that Besov spaces and modulation spaces are rather similar, both of them are the combinations of frequency decomposition operators and function spaces $\ell^q(L^p)$. In fact, we have the following inclusion results.

Theorem 3.6 (Embedding) *Let $0 < p, q \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. We have the following results.*

(1) $B_{p,q}^{s_1} \subset M_{p,q}^{s_2}$ if and only if $s_1 \geq s_2 + \tau(p, q)$, where

$$\tau(p, q) = \max \left\{ 0, n \left(\frac{1}{q} - \frac{1}{p} \right), n \left(\frac{1}{q} + \frac{1}{p} - 1 \right) \right\};$$

(2) $M_{p,q}^{s_1} \subset B_{p,q}^{s_2}$ if and only if $s_1 \geq s_2 + \sigma(p, q)$, where

$$\sigma(p, q) = \max \left\{ 0, n \left(\frac{1}{p} - \frac{1}{q} \right), n \left(1 - \frac{1}{p} - \frac{1}{q} \right) \right\}.$$

The inclusions between Besov and modulation spaces in the cases $(1/p, 1/q) \in [0, 1]^2$ were first discussed by Gröbner [17] and he has never published his results. When $(1/p, 1/q)$ is in the vertices of the square $[0, 1]^2$, Gröbner's results are optimal. Afterwards, Toft [48] obtained the sufficiency of Theorem 3.6 in the cases $(1/p, 1/q) \in [0, 1]^2$. Sugimoto and Tomita [46] showed the necessity of the first inclusion of Theorem 3.6 in the cases $(1/p, 1/q) \in [0, 1]^2$, and by duality the second inclusion is also sharp if $(1/p, 1/q) \in [0, 1]^2$ and $p, q \neq \infty$. Sugimoto and Tomita's idea is to use Feichtinger's norm and the dilation property of modulation spaces. In [54, 55, 56] the authors proved the conclusions of Theorem 3.6 by using frequency-uniform decomposition techniques.

Corollary 3.7 *We have the following inclusions.*

$$B_{2,1}^{s+n/2} \subset M_{2,1}^s \subset B_{2,1}^s, \quad B_{\infty,1}^{s+n} \subset M_{\infty,1}^s \subset B_{\infty,1}^s.$$

The above embedding theorem is of importance for the study of nonlinear PDEs. As for the inclusions between L^p -Sobolev spaces and modulation spaces, it has been explicitly determined by a recent work of Sugimoto and Kobayashi [32]:

Theorem 3.8 *Let $1 \leq p, q \leq \infty$ and $s_1, s_2 \in \mathbb{R}$. Then $H_p^{s_1} \subset M_{p,q}^{s_2}$ if and only if one of the following conditions is satisfied:*

- (1) $q \geq p > 1$, $s_1 \geq s_2 + \tau(p, q)$; (2) $p > q$, $s_1 > s_2 + \tau(p, q)$;
- (3) $p = 1, q = \infty$, $s_1 \geq s_2 + \tau(1, \infty)$; (4) $p = 1, q \neq \infty$, $s_1 > s_2 + \tau(1, q)$,

and $M_{p,q}^{s_1} \subset H_p^{s_2}$ if and only if one of the following conditions is satisfied:

- (1) $q \leq p < \infty$, $s_1 \geq s_2 + \sigma(p, q)$; (2) $p < q$, $s_1 > s_2 + \sigma(p, q)$;
- (3) $p = \infty, q = 1$, $s_1 \geq s_2 + \sigma(\infty, 1)$; (4) $p = \infty, q \neq 1$, $s_1 > s_2 + \sigma(\infty, q)$,

where $\tau(p, q)$ and $\sigma(p, q)$ are the same indices as in Theorem 3.6.

3.3 Dilation property of modulation spaces

Roughly speaking, homogeneous Sobolev spaces and their generalizations including homogeneous Besov and Triebel spaces have scales like

$$\|f(\lambda \cdot)\|_X \sim \lambda^{\theta(X)} \|f\|_X.$$

However, the scaling properties of modulation spaces are very complicated, which is quite different from the classical Sobolev spaces. The following result is due to Sugimoto and Tomita. Let $\tau(p, q)$ and $\sigma(p, q)$ be as in Theorem 3.6. Denote

$$\mu_1(p, q) = \tau(p, q) - n/p, \quad \mu_2(p, q) = -\sigma(p, q) - n/p$$

Theorem 3.9 ([46]) *Let $1 \leq p, q \leq \infty$. The following are true.*

(1) *There exists a constant $C > 0$ such that*

$$C^{-1} \lambda^{\mu_2(p, q)} \|f\|_{M_{p, q}} \leq \|f(\lambda \cdot)\|_{M_{p, q}} \leq C \lambda^{\mu_1(p, q)} \|f\|_{M_{p, q}}$$

for all $f \in M_{p, q}$ and $\lambda \geq 1$. Conversely, if there is a constant $C > 0$ such that

$$C^{-1} \lambda^\beta \|f\|_{M_{p, q}} \leq \|f(\lambda \cdot)\|_{M_{p, q}} \leq C \lambda^\alpha \|f\|_{M_{p, q}}$$

for all $f \in M_{p, q}$ and $\lambda \geq 1$, then $\alpha \geq \mu_1(p, q)$ and $\beta \leq \mu_2(p, q)$.

(2) *There exists a constant $C > 0$ such that*

$$C^{-1} \lambda^{\mu_1(p, q)} \|f\|_{M_{p, q}} \leq \|f(\lambda \cdot)\|_{M_{p, q}} \leq C \lambda^{\mu_2(p, q)} \|f\|_{M_{p, q}}$$

for all $f \in M_{p, q}$ and $0 < \lambda \leq 1$. Conversely, if there is a constant $C > 0$ such that

$$C^{-1} \lambda^\alpha \|f\|_{M_{p, q}} \leq \|f(\lambda \cdot)\|_{M_{p, q}} \leq C \lambda^\beta \|f\|_{M_{p, q}}$$

for all $f \in M_{p, q}$ and $0 < \lambda \leq 1$, then $\alpha \geq \mu_1(p, q)$ and $\beta \leq \mu_2(p, q)$.

4 NLS and NLKG in modulation spaces

As indicated in §2, the dispersive semi-group combined with the frequency-uniform decomposition operator has some advantages and we discuss them in this section. The results of this section can be found in [2, 3, 11, 56, 55].

4.1 Schrödinger and Klein-Gordon semigroup in modulation spaces

Let $S(t) = e^{it\Delta}$ denote the Schrödinger semi-group. In [56], Wang, Zhao and Guo obtained the uniform boundedness for the Ginzburg-Landau semi-group $L(t) = e^{(a+i)t\Delta}$ ($a > 0$) in modulation spaces and their proof is also adapted to the Schrödinger semi-group ($a = 0$ in $L(t)$).

Proposition 4.1 (Uniform boundedness of $S(t)$ in $M_{p,q}^s$) *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $0 < q < \infty$. Then we have*

$$\|S(t)f\|_{M_{p,q}^s} \leq C(1 + |t|)^{n|1/2-1/p|} \|f\|_{M_{p,q}^s}. \quad (4.1)$$

Shortly after the work [56], Proposition 4.1 is independently obtained by Bényi, Gröchenig, Okoudjou and Rogers in [3] and their result contains more general semi-group $e^{it(-\Delta)^\alpha}$ with $\alpha \leq 1$, whose proof is based on the short-time frequency analysis technique. Miyachi, Nicola, Riveti, Taracco and Tomita [34] were able to consider the case $\alpha > 1$, Chen, Fan and Sun [7] obtained some refined estimates for $e^{it(-\Delta)^\alpha}$ with any $\alpha > 0$.

Now we consider the truncated decay of $S(t)$.

Proposition 4.2 *Let $s \in \mathbb{R}$, $2 \leq p < \infty$, $1/p + 1/p' = 1$ and $0 < q < \infty$. Then we have*

$$\|S(t)f\|_{M_{p,q}^s} \leq C(1 + |t|)^{-n(1/2-1/p)} \|f\|_{M_{p',q}^s}. \quad (4.2)$$

Propositions 4.1 and 4.2 answer Questions 2.1 and 2.2, and moreover, they are optimal in the sense that the powers of time variable are sharp, cf. [12]. Now we consider the truncated decay estimate for the Klein-Gordon semi-group $G(t) = e^{it\omega^{1/2}}$ where $\omega = I - \Delta$.

Proposition 4.3 *Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $0 < q < \infty$. Then we have*

$$\|G(t)f\|_{M_{p,q}^s} \leq C(1 + |t|)^{n|1/2-1/p|} \|f\|_{M_{p,q}^s}. \quad (4.3)$$

It is known that $G(t)$ satisfies the following $L^p - L^{p'}$ estimate

$$\|G(t)f\|_{H_p^{-2\sigma(p)}} \leq C|t|^{-n(1/2-1/p)} \|f\|_{p'}, \quad (4.4)$$

where

$$2 \leq p < \infty, \quad 2\sigma(p) = (n+2) \left(\frac{1}{2} - \frac{1}{p} \right). \quad (4.5)$$

From (4.4) it follows that

Proposition 4.4 *Let $s \in \mathbb{R}$, $2 \leq p < \infty$, $1/p + 1/p' = 1$, $0 < q < \infty$, $\theta \in [0, 1]$ and $\sigma(p)$ is as in (4.5). Then we have*

$$\|G(t)f\|_{M_{p,q}^s} \leq C(1 + |t|)^{-n\theta(1/2-1/p)} \|f\|_{M_{p',q}^{s+2\sigma(p)\theta}}. \quad (4.6)$$

4.2 Strichartz estimates in modulation spaces

For convenience, we write

$$\|f\|_{\ell_{\square}^{s,q}(L^{\gamma}(I,L^p))} = \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_{L^{\gamma}(I,L^p)}^q \right)^{1/q}, \quad (4.7)$$

$\ell_{\square}^q(L^{\gamma}(I,L^p)) := \ell_{\square}^{0,q}(L^{\gamma}(I,L^p))$, $\ell_{\square}^q(L_{x,t \in I}^p) := \ell_{\square}^q(L^p(I,L^p))$. Recall that the truncated decay can be generalized to the following estimate

$$\|U(t)f\|_{M_{p,q}^{\alpha}} \leq C(1+|t|)^{-\delta} \|f\|_{M_{p',q}}, \quad (4.8)$$

where $2 \leq p < \infty$, $1 \leq q < \infty$, $\alpha = \alpha(p) \in \mathbb{R}$, $\delta = \delta(p) > 0$, α and δ are independent of $t \in \mathbb{R}$, $U(t)$ is a dispersive semi-group,

$$U(t) = \mathfrak{F}^{-1} e^{itP(\xi)} \mathfrak{F}, \quad (4.9)$$

and $P(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is a real valued function. In the sequel we will assume that $U(t)$ satisfies conditions (4.8) and (4.9), from which we can get some Strichartz inequalities for $U(t)$ in modulation spaces (cf. [55]).

Proposition 4.5 (Strichartz inequalities) *Let $U(t)$ satisfy (4.8) and (4.9). For any $\gamma \geq 2 \vee (2/\delta)$, we have*

$$\|U(t)f\|_{\ell_{\square}^{\alpha/2,q}(L^{\gamma}(\mathbb{R},L^p))} \leq C \|f\|_{M_{2,q}}. \quad (4.10)$$

In addition, if $\gamma \geq q$, then we have

$$\|U(t)f\|_{L^{\gamma}(\mathbb{R},M_{p,q}^{\alpha/2})} \leq C \|f\|_{M_{2,q}}. \quad (4.11)$$

Denote

$$(\mathfrak{U}f)(t) = \int_0^t U(t-s)f(s,\cdot)ds. \quad (4.12)$$

Proposition 4.6 *Let $U(t)$ satisfy (4.8) and (4.9). For any $\gamma \geq 2 \vee (2/\delta)$, we have*

$$\|\mathfrak{U}f\|_{\ell_{\square}^q(L^{\infty}(\mathbb{R},L^2))} \leq C \|f\|_{\ell_{\square}^{-\alpha/2,q}(L^{\gamma'}(\mathbb{R},L^{p'}))}. \quad (4.13)$$

In addition, if $\gamma' \leq q$, then

$$\|\mathfrak{U}f\|_{L^{\infty}(\mathbb{R},M_{2,q})} \leq C \|f\|_{L^{\gamma'}(\mathbb{R},M_{p',q}^{-\alpha/2})}. \quad (4.14)$$

Proposition 4.7 *Assume that $U(t)$ satisfies (4.8) and (4.9), $\gamma \geq \max(2/\delta, 2)$. Then we have*

$$\|\mathfrak{U}f\|_{\ell_{\square}^{\alpha/2,q}(L^{\gamma}(\mathbb{R},L^p))} \leq C \|f\|_{\ell_{\square}^q(L^1(\mathbb{R},L^2))}. \quad (4.15)$$

In addition, if $\gamma \geq q$, then

$$\|\mathfrak{U}f\|_{L^{\gamma}(\mathbb{R},M_{p,q}^{\alpha/2})} \leq C \|f\|_{L^1(\mathbb{R},M_{2,q})}. \quad (4.16)$$

The Schrödinger semi-group corresponds to the cases $\alpha = 0$, $\delta = n(1/2 - 1/p)$ and $2 \leq p < \infty$. Taking $q = 1$ in Propositions 4.5–4.7, we immediately have

Corollary 4.8 *Let $2 \leq p < \infty$, $\gamma \geq 2 \vee \gamma(p)$, and*

$$\frac{2}{\gamma(p)} = n\left(\frac{1}{2} - \frac{1}{p}\right). \quad (4.17)$$

Let $S(t) = e^{it\Delta}$, $\mathfrak{A} = \int_0^t S(t-s) \cdot ds$. Then

$$\|S(t)\varphi\|_{\ell_{\square}^1(L^\gamma(\mathbb{R}, L^p))} \leq C\|\varphi\|_{M_{2,1}}, \quad (4.18)$$

$$\|\mathfrak{A}f\|_{\ell_{\square}^1(L^\gamma(\mathbb{R}, L^p)) \cap \ell_{\square}^1(L^\infty(\mathbb{R}, L^2))} \leq C\|f\|_{\ell_{\square}^1(L^{\gamma'}(\mathbb{R}, L^{p'}))}. \quad (4.19)$$

Similar to Corollary 4.8, we have

Corollary 4.9 *Let $2 \leq p < \infty$, $\theta \in (0, 1]$, $1 \leq q < \infty$,*

$$\frac{2}{\gamma_\theta(p)} = n\theta\left(\frac{1}{2} - \frac{1}{p}\right), \quad 2\sigma = (n+2)\theta\left(\frac{1}{2} - \frac{1}{p}\right). \quad (4.20)$$

Let $G(t)$ be as in (4.4), $\mathfrak{G} = \int_0^t G(t-s) \cdot ds$. Then for any $\gamma \geq 2 \vee \gamma_\theta(p)$, we have

$$\|G(t)\varphi\|_{\ell_{\square}^{-\sigma, q}(L^\gamma(\mathbb{R}, L^p))} \leq C\|\varphi\|_{M_{2,q}}, \quad (4.21)$$

$$\|\mathfrak{G}f\|_{\ell_{\square}^{-\sigma, q}(L^\gamma(\mathbb{R}, L^p)) \cap \ell_{\square}^q(L^\infty(\mathbb{R}, L^2))} \leq C\|f\|_{\ell_{\square}^{\sigma, q}(L^{\gamma'}(\mathbb{R}, L^{p'}))}. \quad (4.22)$$

Related Strichartz estimates in Wiener amalgam spaces for the Schrödinger equation were obtained by Cordero and Nicola [10].

4.3 Wellposedness for NLS and NLKG

We study the Cauchy problem for NLS and give partial answers to Question 2.3. Let us consider

$$iu_t + \Delta u = f(u), \quad u(0, x) = u_0(x). \quad (4.23)$$

Noticing that $B_{\infty,1}^n \subset M_{\infty,1} \subset B_{\infty,1}^0 \subset L^\infty$ are sharp embeddings, up to now, we can not get the wellposedness of NLS in L^∞ or in $B_{\infty,1}^0$. However, we can obtain the local wellposedness of NLS in $M_{\infty,1}$. We have

Theorem 4.10 ([2, 11]) *Let $n \geq 1$, $f(u) = \lambda|u|^\kappa u$, $\kappa \in 2\mathbb{N}$, $\lambda \in \mathbb{R}$, $u_0 \in M_{p,1}$ and $1 \leq p \leq \infty$. Then there exists a $T > 0$ such that (4.23) has a unique solution $u \in C([0, T], M_{p,1})$. Moreover, if $T < \infty$, then $\limsup_{t \nearrow T} \|u(t)\|_{M_{p,1}} = \infty$.*

If the nonlinearity has an exponential growth, say $f(u) = \lambda(e^{|u|^2} - 1)u$, the result in Theorem 4.10 also holds. Noticing that $B_{2,1}^{n/2} \subset M_{2,1} \subset B_{2,1}^0 \cap C(\mathbb{R}^n)$ are sharp embeddings, we can get that NLS is global wellposed in $M_{2,1}$ if initial data are sufficiently small.

Theorem 4.11 ([55]) *Let $n \geq 1$, $f(u) = \lambda|u|^\kappa u$, $\kappa \in 2\mathbb{N}$, $\lambda \in \mathbb{R}$, $\kappa \geq 4/n$, $u_0 \in M_{2,1}$ and there exists a sufficiently small $\delta > 0$ such that $\|u_0\|_{M_{2,1}} \leq \delta$. Then (4.23) has a unique solution*

$$u \in C(\mathbb{R}, M_{2,1}) \cap \ell_{\square}^1(L_{x,t \in \mathbb{R}}^p), \quad (4.24)$$

where $p \in [2 + 4/n, 2 + \kappa] \cap \mathbb{N}$, $\ell_{\square}^1(L_{x,t \in \mathbb{R}}^p)$ is as in (4.7).

Theorem 4.12 ([55]) *Let $n \geq 2$, $f(u) = \lambda(e^{\varrho|u|^2} - 1)u$, $\lambda \in \mathbb{C}$ and $\varrho > 0$. Assume that $u_0 \in M_{2,1}$ and there exists a sufficiently small $\delta > 0$ such that $\|u_0\|_{M_{2,1}} \leq \delta$. Then (4.23) has a unique solution*

$$u \in C(\mathbb{R}, M_{2,1}) \cap \ell_{\square}^1(L_{x,t \in \mathbb{R}}^4). \quad (4.25)$$

We now consider the initial value problem for NLKG,

$$u_{tt} + (I - \Delta)u + f(u) = 0, \quad u(0) = u_0, \quad u_t(0) = u_1. \quad (4.26)$$

Analogous to NLS, we have

Theorem 4.13 ([2, 11]) *Let $n \geq 1$, $f(u) = \lambda|u|^\kappa u$, $\kappa \in 2\mathbb{N}$, $\lambda \in \mathbb{R}$, $(u_0, u_1) \in M_{p,1} \times M_{p,1}^{-1}$ and $1 \leq p \leq \infty$. Then there exists a $T > 0$ such that (4.23) has a unique solution $(u, u_t) \in C([0, T], M_{p,1}) \times C([0, T], M_{p,1}^{-1})$. Moreover, if $T < \infty$, then $\limsup_{t \nearrow T} (\|u(t)\|_{M_{p,1}} + \|u_t(t)\|_{M_{p,1}^{-1}}) = \infty$.*

If the nonlinearity has an exponential growth, the corresponding results as in Theorem 4.13 also hold.

Theorem 4.14 ([55]) *Let $n \geq 1$, $f(u) = \lambda u^{1+\kappa}$, $\kappa \in \mathbb{N}$ and $\kappa \geq 4/n$. Put*

$$\sigma = \frac{n+2}{n(2+\kappa)}. \quad (4.27)$$

Assume that $(u_0, u_1) \in M_{2,1}^\sigma \times M_{2,1}^{\sigma-1}$ and there exists a sufficiently small $\delta > 0$ such that $\|u_0\|_{M_{2,1}^\sigma} + \|u_1\|_{M_{2,1}^{\sigma-1}} \leq \delta$. Then (4.26) has a unique solution

$$u \in C(\mathbb{R}, M_{2,1}^\sigma) \cap \ell_{\square}^1(L_{x,t \in \mathbb{R}}^{2+\kappa}). \quad (4.28)$$

Theorem 4.15 ([55]) *Let $n \geq 2$, $f(u) = \sinh u - u$ and $\sigma = (n+2)/4n$. Assume that $(u_0, u_1) \in M_{2,1}^\sigma \times M_{2,1}^{\sigma-1}$ and there exists a sufficiently small $\delta > 0$ such that $\|u_0\|_{M_{2,1}^\sigma} + \|u_1\|_{M_{2,1}^{\sigma-1}} \leq \delta$. Then (4.26) has a unique solution*

$$u \in C(\mathbb{R}, M_{2,1}^\sigma) \cap \ell_{\square}^1(L_{x,t \in \mathbb{R}}^4). \quad (4.29)$$

5 Derivative nonlinear Schrödinger equations

We study the initial value problem for the derivative nonlinear Schrödinger equation (gDNLS)

$$iu_t + \Delta_{\pm} u = F(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad u(0, x) = u_0(x), \quad (5.1)$$

where u is a complex-valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\Delta_{\pm} u = \sum_{i=1}^n \varepsilon_i \partial_{x_i}^2, \quad \varepsilon_i \in \{1, -1\}, \quad i = 1, \dots, n, \quad (5.2)$$

$\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$, $F : \mathbb{C}^{2n+2} \rightarrow \mathbb{C}$ is a series of $z \in \mathbb{C}^{2n+2}$,

$$F(z) = F(z_1, \dots, z_{2n+2}) = \sum_{3 \leq |\beta| < \infty} c_{\beta} z^{\beta}, \quad c_{\beta} \in \mathbb{C}, \quad (5.3)$$

$|c_{\beta}| \leq C^{|\beta|}$. The typical nonlinear term is

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = |u|^2 \vec{\lambda} \cdot \nabla u + u^2 \vec{\mu} \cdot \nabla \bar{u} + |u|^2 u,$$

see [9, 14, 51]. Another model is

$$F(u, \bar{u}, \nabla u, \nabla \bar{u}) = (1 + |u|^2)^{-1} (\nabla u)^2 \bar{u} = \sum_{k=0}^{\infty} (-1)^k |u|^{2k} (\nabla u)^2 \bar{u}, \quad |u| < 1,$$

which is an equivalent version of the Schrödinger flow [20]. The non-elliptic gDNLS arises in the strongly interacting many-body systems near the criticality, where anisotropic interactions are manifested by the presence of the non-elliptic case, as well as additional residual terms which involve cross derivatives of the independent variables [9, 14, 51]. Some water wave and completely integrable system models in higher spatial dimensions are also non-elliptic, cf. [1, 58, 59]. A large amount of work has been devoted to the study of gDNLS, see [19, 20, 22, 23, 24, 25, 28, 36, 42, 43, 45].

Since the nonlinearity in gDNLS contains derivative terms and the Strichartz inequalities can not absorb any derivatives, gDNLS can not be solved if we use only the Strichartz estimate. One needs to look for some other ways to handle the derivative terms in the nonlinearity. Up to now, three kinds of methods seem to be very useful for gDNLS. One is to use the energy estimate to deal with the derivatives in the nonlinearity, the second way is to use Bourgain's space $X^{s,b}$ and the third technique is Kato's smooth effect estimates. Of course, there are some connections between these methods.

In this survey paper we only discuss the smooth effect estimates together with the frequency-uniform decomposition techniques for gDNLS and we show that it is globally wellposed and scattering in a class of modulation spaces.

For convenience, we denote

$$S(t) = e^{it\Delta_\pm} = \mathfrak{F}^{-1} e^{it \sum_{j=1}^n \varepsilon_j \xi_j^2} \mathfrak{F}, \quad \mathfrak{A}f(t, x) = \int_0^t S(t - \tau) f(\tau, x) d\tau.$$

We now state a global wellposedness and scattering result for gDNLS in modulation spaces. We denote by $L_{x_i}^{p_1} L_{(x_j)_{j \neq i}}^{p_2} L_t^{p_2} := L_{x_i}^{p_1} L_{(x_j)_{j \neq i}}^{p_2} L_t^{p_2}(\mathbb{R}^{1+n})$ the anisotropic Lebesgue space for which the norm is defined by

$$\|f\|_{L_{x_i}^{p_1} L_{(x_j)_{j \neq i}}^{p_2} L_t^{p_2}} = \left\| \|f\|_{L_{x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n}^{p_2} L_t^{p_2}(\mathbb{R} \times \mathbb{R}^{n-1})} \right\|_{L_{x_i}^{p_1}(\mathbb{R})}. \quad (5.4)$$

For $k = (k_1, \dots, k_n)$, we write

$$\begin{aligned} \|u\|_{X_\alpha^s} &= \sum_{i, \ell=1}^n \sum_{k \in \mathbb{Z}^n, |k_i| > 10V \max_{j \neq i} |k_j|} \langle k_i \rangle^{s-1/2} \|\partial_{x_\ell}^\alpha \square_k u\|_{L_{x_i}^\infty L_{(x_j)_{j \neq i}}^2 L_t^2} \\ &\quad + \sum_{i, \ell=1}^n \sum_{k \in \mathbb{Z}^n} \|\partial_{x_\ell}^\alpha \square_k u\|_{L_{x_i}^2 L_{(x_j)_{j \neq i}}^\infty L_t^\infty}, \end{aligned} \quad (5.5)$$

$$\|u\|_{S_\alpha^s} = \sum_{\ell=1}^n \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{s-1} \|\partial_{x_\ell}^\alpha \square_k u\|_{L_t^\infty L_x^2 \cap L_t^3 L_x^6}, \quad (5.6)$$

$$\|u\|_{X^s} = \sum_{\alpha=0,1} \|u\|_{X_\alpha^s}, \quad \|u\|_{S^s} = \sum_{\alpha=0,1} \|u\|_{S_\alpha^s}. \quad (5.7)$$

Theorem 5.1 ([53, 52]) *Let $n \geq 3$, $u_0 \in M_{2,1}^{3/2}$ and there exists a suitably small $\delta > 0$ such that $\|u_0\|_{M_{2,1}^{3/2}} \leq \delta$. Then (5.1) has a unique solution $u \in C(\mathbb{R}, M_{2,1}^{3/2}) \cap X^{3/2} \cap S^{3/2}$, $\|u\|_{X^{3/2} \cap S^{3/2}} \leq C\delta$. Moreover, the scattering operator S of (5.1) carries a whole zero neighborhood in $C(\mathbb{R}, M_{2,1}^{3/2})$ into $C(\mathbb{R}, M_{2,1}^{3/2})$.*

Remark 5.2 *Recently, this technique was also developed for the Navier-Stokes equation and the dissipative nonlinear electrohydrodynamic system [56, 21, 13].*

6 Canonical transformations

It is interesting to generalize results in previous two sections to the case of dispersive operators $a(D)$ instead of $-\Delta$. Recently Ruzhansky and Sugimoto has introduced a new idea to establish fundamental estimates for dispersive equations based on the idea of canonical transformations, and this attempt is quite successful for smoothing estimates ([41]).

Let $\psi : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}^n \setminus 0$ be C^∞ -maps satisfying $\psi(\lambda\xi) = \lambda\psi(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n \setminus 0$, and let

$$\begin{aligned} I_\psi u(x) &= \mathfrak{F}^{-1}[(\mathfrak{F}u)(\psi(\xi))](x) \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x \cdot \xi - y \cdot \psi(\xi))} u(y) dy d\xi. \end{aligned} \quad (6.1)$$

We remark that we have the formula

$$a(D) \cdot I_\psi = I_\psi \cdot \sigma(D), \quad a(\xi) = (\sigma \circ \psi)(\xi). \quad (6.2)$$

For example, for a positive function $a(\xi)$ satisfying $a(\lambda\xi) = \lambda^2 a(\xi)$ for all $\lambda > 0$ and $\xi \in \mathbb{R}^n \setminus 0$, we have

$$a(D) \cdot I_\psi = I_\psi \cdot (-\Delta)$$

if we take

$$\sigma(\eta) = |\eta|^2, \quad \psi(\xi) = \sqrt{a(\xi)} \frac{\nabla a(\xi)}{|\nabla a(\xi)|},$$

provided that $\nabla a(\xi) \neq 0$ for $\xi \neq 0$. The latter is achieved if we assume that the Gaussian curvature of the hypersurface $\Sigma = \{\xi : a(\xi) = 1\}$ never vanishes, in which case also I_ψ has the inverse $I_{\psi^{-1}} = I_\psi^{-1}$ because the Gauss map $\Sigma \ni \xi \mapsto \frac{\nabla a(\xi)}{|\nabla a(\xi)|} \in \mathbb{S}^{n-1}$ is a diffeomorphism.

Thus we can induce the same estimate for $a(D)$ from the estimates for $-\Delta$ if we establish the boundedness of operators I_ψ and $I_{\psi^{-1}}$ on modulation spaces. Ruzhansky, Sugimoto, Toft and Tomita [40] discuss such boundedness properties, and we have a positive result for the local boundedness. Let L_s^q be the space of functions such that $\langle x \rangle^s f \in L^q$:

Theorem 6.1 *Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, and let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be such that the pullback $\psi^* : f \mapsto f \circ \psi$ is bounded on $L_s^q(\mathbb{R}^n)$. Then I_ψ is locally bounded on $M_{p,q}^s(\mathbb{R}^n)$.*

By Theorem 6.1 and Propositions 4.1, 4.2, we straightforwardly obtain estimates (4.1) and (4.2) for $S(t) = \chi(x) e^{-it a(D)} \tilde{\chi}(x)$ (and we have to also restrict the case to $1 \leq q \leq \infty$) from the estimates for $S(t) = e^{it\Delta}$, where $\chi, \tilde{\chi}$ are cut-off functions.

As for the global boundedness of I_φ , we have unfortunately a negative result:

Theorem 6.2 *Let $1 \leq p, q \leq \infty$, $2 \neq p < \infty$, and let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 -function. Assume that operator I_ψ is bounded on $M_{p,q}(\mathbb{R}^n)$. Then ψ is an affine mapping.*

7 Open questions

It is known that the algebra property of function spaces is of importance for PDE. Up to now the following question is not clear for us:

Question 7.1 *Let $\alpha \in (0, \infty)$. Does*

$$\| |u|^\alpha u \|_{M_{p,1}} \leq C \|u\|_{M_{p,1}}^{\alpha+1}$$

hold for all $u \in M_{p,1}$?

It is known that if $\alpha \in 2\mathbb{N}$, the answer is affirmative. If α is not an even integer, the question seems very difficult, cf. [47].

The global well posedness of NLS in modulation spaces for large initial data seems open.

Question 7.2 *Can we show that NLS (2.5) is global well posed if the initial data $u_0 \in M_{2,1}$ is large?*

We should also discuss non-affine transforms which induce the globally bounded canonical transformations. Note that such transforms must not be C^1 -mappings in view of Theorem 6.2.

Question 7.3 *Can we show the global boundedness of the operator I_ψ in (6.1) on modulation spaces for a homogeneous change of variables ψ ?*

Some partial answers to this question appeared in [40].

References

- [1] M. J. Ablowitz, R. Haberman, Nonlinear evolution equations in two and three dimensions, *Phys. Rev. Lett.*, **35** (1975), 1185–1188.
- [2] A. Bényi, K.A. Okoudjou, Local well-posedness of nonlinear dispersive equations on modulation spaces, *Bull. London Math. Soc.*, **41** (2009), 549–558.
- [3] A. Bényi, K. Gröchenig, K.A. Okoudjou and L.G. Rogers, Unimodular Fourier multiplier for modulation spaces, *J. Funct. Anal.*, **246** (2007), 366–384.
- [4] J. Bergh and J. Löfström, *Interpolation Spaces*, Springer–Verlag, 1976.
- [5] T. Cazenave and F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s , *Nonlinear Anal. TMA*, **14** (1990), 807–836.
- [6] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, Vol. **10**, 2003.
- [7] J. C. Chen, D. S. Fan and L. Sun, Asymptotic estimates For unimodular Fourier multipliers on modulation spaces, Preprint.
- [8] J. Colliander, *Nonlinear Schrödinger equations*, Lecture notes, 2004.
- [9] P. A. Clarkson and J. A. Tuszyrski, Exact solutions of the multidimensional derivative nonlinear Schrödinger equation for many-body systems near criticality, *J. Phys. A: Math. Gen.* **23** (1990), 4269–4288.
- [10] E. Cordero, F. Nicola, Some new Strichartz estimates for the Schrödinger equation. *J. Differential Equations*, **245** (2008), 1945–1974.
- [11] E. Cordero, F. Nicola, Remarks on Fourier multipliers and applications to the wave equation, *J. Math. Anal. Appl.*, **353** (2009), 583–591.

- [12] E. Cordero, F. Nicola, Sharpness of some properties of Wiener amalgam and modulation spaces. *Bull. Aust. Math. Soc.*, **80** (2009), 105–116.
- [13] C. Deng, J. H. Zhao, S. B. Cui, Well-posedness of a dissipative nonlinear electrohydrodynamic system in modulation spaces, *Nonlinear Anal., TMA*, **73** (2010), 2088–2100.
- [14] J. M. Dixon and J. A. Tuszynski, Coherent structures in strongly interacting many-body systems: II, Classical solutions and quantum fluctuations, *J. Phys. A: Math. Gen.*, **22** (1989), 4895–4920.
- [15] H. G. Feichtinger, Modulation spaces on locally compact Abelian group, Technical Report, University of Vienna, 1983. Published in: “Proc. Internat. Conf. on Wavelet and Applications”, 99–140. New Delhi Allied Publishers, India, 2003.
- [16] L. Grafakos, *Classical and modern Fourier analysis*, Pearson/Prentice Hall, 2004.
- [17] P. Gröbner, *Banachräume Glatter Funktionen und Zerlegungsmethoden*, Doctoral thesis, University of Vienna, 1992.
- [18] K. Gröchenig, *Foundations of Time–Frequency Analysis*, Birkhäuser, Boston, MA, 2001.
- [19] N. Hayashi and T. Ozawa, Finite energy solutions of nonlinear Schrödinger equations of derivative type, *SIAM J. Math. Anal.*, **25** (1994), 1488–1503.
- [20] A. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, II: Global well posedness in dimensions $d \geq 3$, *Commun. Math. Phys.*, **271** (2007), 523–559.
- [21] T. Iwabuchi, Navier-Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices. *J. Differential Equations*, **248** (2010), 1972–2002.
- [22] C. E. Kenig, G. Ponce, C. Rolvent, L. Vega, The genreal quasilinear untrahyperbolic Schrodinger equation, *Adv. Math.*, **206** (2006), 402–433.
- [23] C. E. Kenig, G. Ponce, L. Vega, Small solutions to nonlinear Schrödinger equation, *Ann. Inst. Henri Poincaré, Sect C*, **10** (1993), 255–288.
- [24] C. E. Kenig, G. Ponce and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, *Invent. Math.*, **134** (1998), 489–545.
- [25] S. Klainerman, Long-time behavior of solutions to nonlinear evolution equations, *Arch. Rational Mech. Anal.*, **78** (1982), 73–98.

- [26] S. Klainerman and M. Machedon, Space-time estimates for null forms and the local existence theorem, *Comm. Pure Appl. Math.*, **46** (1993), no. 9, 1221–1268.
- [27] S. Klainerman and M. Machedon, Smoothing estimates for null forms and applications, *Duke Math. J.*, **81** (1995), 99–133.
- [28] S. Klainerman, G. Ponce, Global small amplitude solutions to nonlinear evolution equations, *Commun. Pure Appl. Math.*, **36** (1983), 133–141.
- [29] M. Kobayashi, Modulation spaces $M^{p,q}$ for $0 < p, q \leq \infty$, *J. of Funct. Spaces and Appl.*, **4** (2006), no.3, 329–341.
- [30] M. Kobayashi, Dual of modulation spaces, *J. of Funct. Spaces and Appl.*, **5** (2007), 1–8.
- [31] M. Kobayashi, Y. Sawano, Molecular decomposition of the modulation spaces, *Osaka J. Math.*, **47** (2010), 1029–1053.
- [32] M. Kobayashi, M. Sugimoto, The inclusion relation between Sobolev and modulation spaces, *J. Funct. Anal.* **260** (2011), 3189–3208.
- [33] F. Linares and G. Ponce, On the Davey–Stewartson systems, *Ann. Inst. H. Poincaré, Anal. NonLinéaire*, **10** (1993), 523–548.
- [34] A. Miyachi, F. Nicola, S. Riveti, A. Taracco and N. Tomita, Estimates for unimodular Fourier multipliers on modulation spaces, *Proc. Amer. Math. Soc.*, **137** (2009), 3869–3883.
- [35] T. Ozawa, On the nonlinear Schrödinger equations of derivative type, *Indiana Univ. Math. J.*, **45** (1996), 137–163.
- [36] T. Ozawa and J. Zhai, Global existence of small classical solutions to nonlinear Schrödinger equations, *Ann. I. H. Poincaré, Anal. Non Linéaire*, **25** (2008), 303–311.
- [37] M. Ruzhansky, On the sharpness of Seeger-Sogge-Stein orders, *Hokkaido Math. J.*, **28** (1999), 357–362.
- [38] M. V. Ruzhansky, Singularities of affine fibrations in the regularity theory of Fourier integral operators, *Russian Math. Surveys*, **55** (2000), 93–161.
- [39] M. Ruzhansky, J. Smith, *Dispersive and Strichartz estimates for hyperbolic equations with constant coefficients*, *MSJ Memoirs*, **22**, Mathematical Society of Japan, Tokyo, 2010.

- [40] M. Ruzhansky, M. Sugimoto, J. Toft and N. Tomita *Changes of variables in modulation and Wiener amalgam spaces*, to appear in Math. Nachr.
- [41] M. Ruzhansky and M. Sugimoto, *Smoothing properties of evolution equations via canonical transforms and comparison principle*, to appear in Proc. London Math. Soc.
- [42] M. Ruzhansky and M. Sugimoto, *Structural resolvent estimates and derivative nonlinear Schrödinger equations*, arXiv:1101.5026v1.
- [43] J. Shatah, Global existence of small classical solutions to nonlinear evolution equations, J. Differential Equations, **46** (1982), 409–423.
- [44] J. Shatah, Normal forms and quadratic nonlinear Klein–Gordon equations, Comm. Pure Appl. Math., **38** (1985), 685–696.
- [45] M. Sugimoto, M. Ruzhansky, A smoothing property of Schrödinger equations and a global existence result for derivative nonlinear equations, Advances in analysis, 315–320, World Sci. Publ., Hackensack, NJ, 2005.
- [46] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, J. Funct. Anal., **248** (2007), 79–106.
- [47] M. Sugimoto; N. Tomita; B. X. Wang, Remarks on nonlinear operations on modulation spaces, Integral Transforms and Special Functions, **22** (2011), 351–358
- [48] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, I, J. Funct. Anal., **207** (2004), 399–429.
- [49] H. Triebel, *Theory of Function Spaces*, Birkhäuser–Verlag, 1983.
- [50] H. Triebel, Modulation spaces on the Euclidean n -spaces, Z. Anal. Anwendungen, **2** (1983), 443–457.
- [51] J. A. Tuszynski and J. M. Dixon, Coherent structures in strongly interacting many-body systems: I, Derivation of dynamics, J. Phys. A: Math. Gen., **22** (1989), 4877–4894.
- [52] B. X. Wang, Sharp global well-posedness for non-elliptic derivative Schrödinger equations with small rough data, arXiv:1012.0370.
- [53] B. X. Wang, L. J. Han, C. Y. Huang, Global well-Posedness and scattering for the derivative nonlinear Schrödinger equation with small rough data, Ann. I. H. Poincaré, Anal. Non Linéaire, **26** (2009), 2253–2281.

- [54] B. X. Wang and C. Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, J. Differential Equations, **239** (2007), 213–250.
- [55] B. X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, J. Differential Equations, **231** (2007), 36–73.
- [56] B. X. Wang, L. F. Zhao, B. L. Guo, Isometric decomposition operators, function spaces $E_{p,q}^\lambda$ and their applications to nonlinear evolution equations, J. Funct. Anal., **233** (2006), 1–39.
- [57] N. Wiener, Tauberian theorems, Ann. of Math., **33** (1932), 1–100.
- [58] V. E. Zakharov, E. A. Kuznetson, Multi-scale expansions in the theory of systems integrable by inverse scattering method, Physica D, **18** (1986), 455–463.
- [59] V. E. Zakharov, E. I. Schulman, Degenerated dispersion laws, motion invariant and kinetic equations, Physica D, **1** (1980), 185–250.